

# NONLINEAR SCHRÖDINGER-HELMHOLTZ EQUATION AS NUMERICAL REGULARIZATION OF THE NONLINEAR SCHRÖDINGER EQUATION

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**ABSTRACT.** A regularized  $\alpha$ -system of the Nonlinear Schrödinger Equation (NLS) with  $2\sigma$  nonlinear power in dimension  $N$  is studied. We prove existence and uniqueness of local solution in the case  $1 \leq \sigma < \frac{4}{N-2}$  and existence and uniqueness of global solution in the case  $1 \leq \sigma < \frac{4}{N}$ . When  $\alpha \rightarrow 0^+$ , this regularized system will converge to the classical NLS in the appropriate range. In particular, the purpose of this numerical regularization is to shed light on the profile of the blow up solutions of the original Nonlinear Schrödinger Equation in the range  $\frac{2}{N} \leq \sigma < \frac{4}{N}$ , and in particular for the critical case  $\sigma = \frac{2}{N}$ .

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## 1. INTRODUCTION

The Nonlinear Schrödinger equation (NLS):

$$\begin{aligned} iv_t + \Delta v + |v|^{2\sigma} v &= 0, & x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \\ v(0) &= v_0, \end{aligned} \tag{1}$$

where  $v$  is a complex-valued function in  $\mathbb{R}^N \times \mathbb{R}$ , arises in various physical contexts describing wave propagation in nonlinear media (see, e.g., [14], [21], [22] and [26]). For example, when  $\sigma = 1$ , equation (1) describes propagation of a laser beam in a nonlinear optical medium whose index of refraction is proportional to the wave intensity. Also, the Nonlinear Schrödinger Equation successfully models other wave phenomena such as water waves at the free surface of an ideal fluid as well as plasma waves. In all cases, it is interesting to note that Eq. (1) describes wave propagation in nonhomogeneous linear media with self-induced potential given by  $|v|^{2\sigma}$ .

As it is mentioned above, the  $\sigma = 1$  case is particularly interesting for laser beam propagation in optical Kerr media. Depending on the dimensionality of the space upon which the beam is propagating in, the wave dynamics can be either “simple” or “intricate”. In one space dimension, the NLS equation is known to be integrable and possesses soliton solution that preserves their structure upon collision [1]. The picture in two-dimensional (2D) space is totally different. The 2D NLS equation is not integrable, hence no exact soliton solutions are known. Instead, the 2D NLS equation admits the waveguide solution (also known as Townes soliton)  $v(x, y, t) = R(r) \exp(it)$  with  $r = \sqrt{x^2 + y^2}$  where  $R > 0$  satisfies the nonlinear boundary value problem

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - R + R^3 = 0, \quad \frac{dR}{dr}(0) = 0, \quad \lim_{r \rightarrow +\infty} R(r) = 0. \tag{2}$$

Importantly, the  $L^2$  norm (or power in optics) of the Townes soliton defines a critical value for blow up. If initially the beam's power is larger than that of the Townes soliton,  $\|v_0\|_{L^2}^2 > \|R\|_{L^2}^2$ , then the beam undergoes a finite time blow up. If on the other hand  $\|v_0\|_{L^2}^2 < \|R\|_{L^2}^2$ , then the wave will diffract. Various mechanisms to arrest collapse have been suggested such as nonparaxiality [6], or higher order dispersion [7].

As a result, an important issue that arises in the mathematical study of the NLS is the question of local and global existence of solutions, their uniqueness, as well as the profile of blow up solutions. Knowing answers to such questions may have some consequences on possible physical observations of phenomenon governed by the NLS and in validating its derivation. In the works of Ginibre and Velo [10] and Weinstein [24], it is proved that equation (1) has a unique global solution when  $0 < \sigma < \frac{2}{N}$ , and that it has a unique global solution for "small" initial data for the critical case  $\sigma = \frac{2}{N}$ . The proof of global existence uses the fact that the energy  $\mathcal{N}(v) = \int_{\mathbb{R}^N} |v(x, t)|^2 dx$  and the Hamiltonian  $\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{|v(x, t)|^{2\sigma+2}}{\sigma+1} \right) dx$  are conserved quantities of the dynamics of (1). In the case of  $\sigma \geq \frac{2}{N}$ , Glassey [11] proved that there exist solutions that develop singularities in finite time. In recent years there was an intensive remarkable computational work concerning the blow up for the critical case  $\sigma = \frac{2}{N}$ . For instance, Merle and Raphael have obtained a sharp lower bound on the blow up rate for the  $L^2$  norm of the NLS in  $\mathbb{R}^N$  (see [15] and references therein). Moreover, Fibich and Merle [8] studied self-focusing in bounded domains using a combination of rigorous, asymptotic and numerical results.

Instead of the potential  $|v|^{2\sigma}$ , physicists consider self-gravitational potential (see, e.g., [18], [20]) and come to a new system: Schrödinger-Newton equation (SN):

$$\begin{aligned} iv_t + \Delta v + \psi v &= 0, & x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \\ -\alpha^2 \Delta \psi &= |v|^2, \\ v(0) &= v_0, \end{aligned} \tag{3}$$

where  $\alpha > 0$  is a real constant. System (3) is a Hamiltonian system with the corresponding Hamiltonian  $\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{\psi(x, t)|v(x, t)|^2}{2} \right) dx$  and can be obtained formally by the variational principle  $i \frac{\partial v}{\partial t} = \frac{\delta \mathcal{H}(v)}{\delta v^*}$ , where  $v^*$  denotes the complex conjugate of  $v$ . System (3), or at least its stationary state, has been studied [17], [23], [19]. This coupled system of equations consists of the Schrödinger equation for a wave function  $v$  moving in a potential  $\psi$ , where  $\psi$  is obtained by solving the Poisson equation with source  $\rho = |v|^2$ . It can be thought of as the Schrödinger equation for a particle moving in its own gravitational field. As in the NLS, the energy  $\mathcal{N}(v) = \int_{\mathbb{R}^N} |v(x, t)|^2 dx$  and the Hamiltonian  $\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{\psi(x, t)|v(x, t)|^2}{2} \right) dx$  are also conserved in this system. The question of existence and uniqueness of local and global solutions for system (3) has not been answered completely yet.

Inspired by the  $\alpha$ -models of turbulence (see, e.g., [3], [4], [5], [9], [12], [13] and references therein), we introduce a generalization of (3), the Schrödinger-Helmholtz (SH) regularization of the classical NLS:

$$\begin{aligned} iv_t + \Delta v + u|v|^{\sigma-1}v &= 0, & x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \\ u - \alpha^2 \Delta u &= |v|^{\sigma+1}, \\ v(0) &= v_0, \end{aligned} \tag{4}$$

where  $\alpha > 0$  and  $\sigma \geq 1$ . System (4) is a Hamiltonian system with the corresponding Hamiltonian  $\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{u(x, t)|v(x, t)|^{\sigma+1}}{\sigma+1} \right) dx$  and can be obtained formally by the variational principle  $i \frac{\partial v}{\partial t} = \frac{\delta \mathcal{H}(v)}{\delta v^*}$ , where again  $v^*$  denotes the complex conjugate of  $v$ . In this system, we can regard the wave function  $v$  moves in a potential  $u|v|^{\sigma-1}$ , where  $u$  is obtained by solving the Helmholtz elliptic

problem  $u - \alpha^2 \Delta u = |v|^{\sigma+1}$ . Observe that the energy  $\mathcal{N}(v) = \int_{\mathbb{R}^N} |v(x, t)|^2 dx$  and the Hamiltonian  $\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{u(x, t)|v(x, t)|^{\sigma+1}}{\sigma+1} \right) dx$  are conserved in this system. When  $\sigma = 1$ , we have the potential  $u$  as in the SN with the only difference that the Poisson equation is modified as a Helmholtz equation. So we consider this system as a generalized system of SN. A more important fact is that when  $\alpha = 0$ , one recovers the classical NLS, therefore we regard this system as a regularization of the classical NLS. In this paper we focus on the case  $\alpha > 0$  and in our subsequential works, we will investigate the behavior when  $\alpha \rightarrow 0^+$ . In particular, we will investigate the case  $\sigma = \frac{2}{N}$ , which is not completely understood.

In this paper, we will study the question of local and global existence of unique solution for system (4). Specifically, we will prove the short time existence of unique solution, when  $1 \leq \sigma < \frac{4}{N-2}$  (we define once and for all  $\frac{4}{N-2} = \infty$  when  $N \leq 2$ ). Moreover, we will show global existence of unique solution when  $1 \leq \sigma < \frac{4}{N}$ . The proof will follow the ideas of [10] and [24] and use the important fact of the conservation of the corresponding energy and the Hamiltonian of (4). All the proofs presented here will apply directly to system (3) as well. So simultaneously we have the same results for system (3): we have short time existence of unique solution when  $1 \leq \sigma < \frac{4}{N-2}$ , and global existence of unique solution when  $1 \leq \sigma < \frac{4}{N}$ . Comparing to the results of the classical NLS (1) ( $\sigma < \frac{2}{N-2}$  for local existence and  $\sigma < \frac{2}{N}$  for global existence), one expects these “better” results for (3) and (4) since the nonlinear terms in (3) and (4) are milder than that of the classical NLS (1). The parametre  $\alpha$  plays an important role in our proofs. In a subsequential paper, we will investigate numerically the blow up profiles of the NLS, in the relevant range of  $\sigma$ , when  $\alpha \rightarrow 0^+$ .

In section 2, we will introduce some essential notations and definitions, and some preliminary results that will be used throughout the paper. Following the work of Ginibre and Velo [10], we prove in section 3 local (in time) existence and uniqueness of solution for system (4) using the contraction mapping principle. In section 4, we will extend the local solution to global existence, for  $1 \leq \sigma < \frac{4}{N}$ , after establishing the required *a priori* estimates for the  $H^1$  norm of the solution, which remains finite for every finite interval of time.

## 2. NOTATIONS AND PRELIMINARIES

In this section we introduce some preliminary results and the basic notations and definitions that will be used throughout this paper.

We denote by  $\|\cdot\|_p$  the norm in the space  $L^p = L^p(\mathbb{R}^N)$  ( $1 \leq p \leq \infty$ ), except for  $p = 2$  where the subscript 2 will be omitted. We will denote by  $\langle \cdot, \cdot \rangle$  the scalar product in  $L^2$ . The conjugate pair  $p, p'$  satisfies the relation  $\frac{1}{p} + \frac{1}{p'} = 1$ . For any real number  $l$ , we denote by  $H^l = H^l(\mathbb{R}^N)$ , the usual Sobolev space. Of special interest is the  $H^1$  Sobolev space with the norm defined by

$$\|v\|_{H^1}^2 = \int_{\mathbb{R}^N} (1 + |\xi|^2) |\hat{v}(\xi)|^2 d\xi, \quad (5)$$

or equivalently,

$$\|v\|_{H^1}^2 = \|v\|^2 + \|\nabla v\|^2. \quad (6)$$

We denote by  $\|u\|_{W^{k,p}} = \left( \sum_{|\alpha| \leq k} \int_{\mathbb{R}^N} |D^\alpha u|^p dx \right)^{1/p}$ ,  $1 \leq p < \infty$ , for  $u$  belongs to the Sobolev space  $W^{k,p}(\mathbb{R}^N)$ . For any interval  $I$  of the real line  $\mathbb{R}$ , and for any Banach space  $\mathcal{B}$ , we denote by  $\mathcal{C}(I, \mathcal{B})$  (respectively  $\mathcal{C}_b(I, \mathcal{B})$ ) the space of continuous (respectively bounded continuous) functions from  $I$  into  $\mathcal{B}$ .

In this paper  $C$  and  $C_\alpha$  will denote constants which might depend on various parameters of the problem.

They might vary in value from one time to another, but they are independent of the solution. When it is relevant we will comment on the asymptotic behavior of these constants as they depend on the corresponding parameters.

First, we recall some classical Gagliardo-Nirenberg and Sobolev inequalities (see, e.g., [2]).

**Proposition 1.** (1) For any  $N \geq 1$ , we have

$$\|v\|_q \leq C\|v\|^{1-\frac{q-2}{2q}N}\|\nabla v\|^{\frac{q-2}{2q}N} \quad \text{for every } v \in H^1, 0 < \frac{q-2}{2q}N \leq 1 \quad (7)$$

$$\|v\|_q \leq C\|v\|_{W^{2,m}} \quad \text{for every } v \in W^{2,m}, q \geq m, 2m > N \quad (8)$$

$$\|v\|_q \leq C\|v\|_{W^{2,m}} \quad \text{for every } v \in W^{2,m}, \frac{1}{q} \geq \frac{1}{m} - \frac{2}{N} \geq 0, q < \infty \quad (9)$$

$$\text{In particular,} \quad (10)$$

$$\|v\|_q \leq C\|v\|_{W^{2,2}} = C\|v\|_{H^2} \quad \text{for every } v \in H^2, 2 \leq q \leq \infty, N \leq 3. \quad (11)$$

(2) For  $N \leq 2$ ,

$$\|v\|_q \leq C\|v\|_{H^1} \quad \text{for every } v \in H^1, 2 \leq q < \infty. \quad (12)$$

With these inequalities at hand, we can process the nonlinear term. Let us rewrite the term

$$f(v) = u|v|^{\sigma-1}v = B(|v|^{\sigma+1})|v|^{\sigma-1}v, \quad (13)$$

where  $B = (I - \alpha^2 \Delta)^{-1}$ , the inverse of the Helmholtz operator. Then  $f$  is a locally Lipschitz mapping from  $H^1$  into  $L^{r'}$ , for some  $r \in (2, \frac{2N}{N-2}]$ , where  $\frac{1}{r} + \frac{1}{r'} = 1$ .

**Proposition 2.** Let  $N \geq 1$  and  $1 \leq \sigma < \frac{4}{N-2}$ . For every  $v_1, v_2 \in H^1 \subset L^r$ , where  $r$  depends on the given  $\sigma$  and belongs to the range  $r \in (2, \frac{2N}{N-2}]$  (we consider  $\frac{2N}{N-2}$  as  $\infty$  when  $N \leq 2$ ), we have  $\|f(v_1) - f(v_2)\|_{r'} \leq k\|v_1 - v_2\|_r$ , where  $k = C_\alpha (\|v_1\|_{H^1} + \|v_2\|_{H^1})^{2\sigma}$  and  $\frac{1}{r} + \frac{1}{r'} = 1$ , for some constant  $C_\alpha$ .

Before we prove this proposition, we will state the following Lemmas:

**Lemma 3.** Let  $N \geq 1$  and  $1 \leq \sigma < \frac{4}{N-2}$ . For every  $v_1, v_2, v \in H^1 \subset L^r$ , where  $r$  depends on the given  $\sigma$  and belongs to the range  $r \in (2, \frac{2N}{N-2}]$ , we have

$$\|B(|v|^\sigma)|v|^{\sigma-1}v(v_1 - v_2)\|_{r'} \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r, \quad (14)$$

$$\|B(|v|^{\sigma-1}v)|v|^{\sigma-1}v(v_1 - v_2)\|_{r'} \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r. \quad (15)$$

*Proof.* First, denote

$$I_1 = \|B(|v|^\sigma)|v|^{\sigma-1}v(v_1 - v_2)\|_{r'}.$$

**Case 1.**  $N \leq 2$ :

By Hölder's inequality, we have

$$\begin{aligned} I_1 &\leq \|B(|v|^\sigma)|v|^{\sigma-1}v\|_{r'\beta_1} \|v_1 - v_2\|_{r'\gamma_1} \\ &= \|B(|v|^\sigma)|v|^{\sigma-1}v\|_{\frac{r}{r-2}} \|v_1 - v_2\|_r, \end{aligned} \quad (16)$$

where in the last equality, we choose  $\gamma_1 = r - 1 > 1$  and  $\frac{1}{\beta_1} + \frac{1}{\gamma_1} = 1$  such that  $r'\gamma_1 = r$  and  $r'\beta_1 = \frac{r}{r-2}$ .

By Cauchy-Schwarz inequality, we have

$$\|B(|v|^\sigma)|v|^{\sigma-1}v\|_{\frac{r}{r-2}} \leq \|B(|v|^\sigma)\|_{\frac{2r}{r-2}} \|v\|_{\frac{2r}{r-2}}. \quad (17)$$

Now, for the elliptic equation  $u - \alpha^2 \Delta u = f$  in  $\mathbb{R}^N$ , we have the regularity property [16], [25]

$$\|u\|_{W^{2,p}} \leq C_\alpha \|f\|_p \quad \text{for any } 1 < p < \infty, \quad (18)$$

where  $C_\alpha$  depends on  $N, p$  and  $\alpha$ , and  $C_\alpha \sim \frac{1}{\alpha^2}$  as  $\alpha \rightarrow 0^+$ . Moreover, for  $\alpha$  fixed,  $C_\alpha \sim p$  as  $p \rightarrow \infty$ . Since  $\frac{2r}{r-2} > 2$ , by (11) and (18), we have

$$\begin{aligned} \|B(|v|^\sigma)\|_{\frac{2r}{r-2}} &\leq C \|B(|v|^\sigma)\|_{W^{2,2}} \\ &\leq C_\alpha \| |v|^\sigma \| \\ &= C_\alpha \|v\|_{2\sigma}^\sigma, \end{aligned}$$

and

$$\| |v|^\sigma \|_{\frac{2r}{r-2}} = \|v\|_{\frac{2r}{r-2}\sigma}^\sigma.$$

Since  $\sigma \geq 1$ , combining the above two terms and applying (12), we have

$$I_1 \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r.$$

**Case 2.  $N \geq 3$ :**

Applying Hölder's inequality, we have

$$\begin{aligned} I_1 &\leq \|B(|v|^\sigma)\|_{r'\theta_1} \| |v|^{\sigma-1} v \|_{r'\beta_1} \|v_1 - v_2\|_{r'\gamma_1} \\ &= \|B(|v|^\sigma)\|_{r'\theta_1} \|v\|_{\sigma r'\beta_1}^\sigma \|v_1 - v_2\|_{r'\gamma_1}, \end{aligned}$$

where  $\frac{1}{\theta_1} + \frac{1}{\beta_1} + \frac{1}{\gamma_1} = 1$ .

Now by (8), (9) and (18), we have

$$\begin{aligned} \|B(|v|^\sigma)\|_{r'\theta_1} &\leq C \|B(|v|^\sigma)\|_{W^{2,m}} \\ &\leq C_\alpha \| |v|^\sigma \|_m \\ &= C_\alpha \|v\|_{\sigma m}^\sigma, \end{aligned}$$

where we require  $\frac{1}{r'\theta_1} \geq \frac{1}{m} - \frac{2}{N}$  when  $\frac{1}{m} - \frac{2}{N} \geq 0$ , or  $r'\theta_1 \geq m$  when  $\frac{1}{m} - \frac{2}{N} < 0$ , for  $m > 1$  to be determined later.

Therefore, we obtain

$$I_1 \leq C_\alpha \|v\|_{\sigma m}^\sigma \|v\|_{\sigma r'\beta_1}^\sigma \|v_1 - v_2\|_{r'\gamma_1}. \quad (19)$$

Now, by requiring  $\sigma m = \sigma r'\beta_1 = r'\gamma_1 = r$ , we have

$$\begin{aligned} \theta_1 &= \frac{r-1}{r-\sigma-2} > 1 \Rightarrow \sigma < r-2 \\ \beta_1 &= \frac{r-1}{\sigma} > 1 \Rightarrow \sigma < r-1 \\ \gamma_1 &= r-1 > 1 \\ m &= \frac{r}{\sigma} > 1 \\ \sigma &< \frac{N+2}{2N}r-1 \quad \text{or} \quad \sigma > \frac{r-2}{2}. \end{aligned}$$

Since  $2 < r \leq \frac{2N}{N-2}$ , we conclude that  $\sigma < \frac{4}{N-2}$ , i.e.,

$$I_1 \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r.$$

By exactly the same steps, inequality (15) follows readily.  $\square$

**Lemma 4.** *Let  $N \geq 1$  and  $1 \leq \sigma < \frac{4}{N-2}$ . For every  $v_1, v_2, v \in H^1 \subset L^r$ , where  $r$  depends on the given  $\sigma$  and belongs to the range  $r \in (2, \frac{2N}{N-2}]$ , we have*

$$\|B(|v|^{\sigma+1})|v|^{\sigma-1}(v_1 - v_2)\|_{r'} \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r, \quad (20)$$

$$\|B(|v|^{\sigma+1})|v|^{\sigma-3}v^2(v_1 - v_2)\|_{r'} \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r. \quad (21)$$

*Proof.* Denote

$$I_2 = \|B(|v|^{\sigma+1})|v|^{\sigma-1}(v_1 - v_2)\|_{r'}.$$

**Case 1.**  $N \leq 2$ :

By Hölder's inequality, we obtain

$$\begin{aligned} I_2 &\leq \|B(|v|^{\sigma+1})|v|^{\sigma-1}\|_{r'\beta_1} \|v_1 - v_2\|_{r'\gamma_1} \\ &= \|B(|v|^{\sigma+1})|v|^{\sigma-1}\|_{\frac{r}{r-2}} \|v_1 - v_2\|_r, \end{aligned}$$

where we choose the same  $\beta_1, \gamma_1$  as in (16).

Now, when  $\sigma > 1$ , again by Hölder's inequality, we have

$$\|B(|v|^{\sigma+1})|v|^{\sigma-1}\|_{\frac{r}{r-2}} \leq \|B(|v|^{\sigma+1})\|_{\frac{r}{r-2}\beta_2} \|v|^{\sigma-1}\|_{\frac{r}{r-2}\gamma_2} \quad (22)$$

$$= \|B(|v|^{\sigma+1})\|_{\frac{r}{r-2}\beta_2} \|v\|_{(\sigma-1)\frac{r}{r-2}\gamma_2}^{\sigma-1}. \quad (23)$$

By choosing  $1 < \beta_2 < \sigma$  and  $2 < r \leq 4$ , one can easily verify that  $\frac{r}{r-2}\beta_2 \geq 2$  and  $(\sigma-1)\frac{r}{r-2}\gamma_2 > 2$ .

By (11) and (18), we obtain

$$\begin{aligned} \|B(|v|^{\sigma+1})\|_{\frac{r}{r-2}\beta_2} &\leq C \|B(|v|^{\sigma+1})\|_{W^{2,2}} \\ &\leq C_\alpha \|v\|^{\sigma+1} \\ &= C_\alpha \|v\|_{2(\sigma+1)}^{\sigma+1}. \end{aligned} \quad (24)$$

Since  $2(\sigma+1) > 2$  and  $(\sigma-1)\frac{r}{r-2}\gamma_2 > 2$  when  $\sigma > 1$ , by (12) we conclude

$$I_2 \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r. \quad (25)$$

Now, when  $\sigma = 1$ , by choosing  $2 < r \leq 4$  in (23) and applying inequality (24) with  $\beta_2 = 1$ , we have

$$\begin{aligned} \|B(|v|^{\sigma+1})|v|^{\sigma-1}\|_{\frac{r}{r-2}} &= \|B(|v|^{\sigma+1})\|_{\frac{r}{r-2}} \\ &\leq C_\alpha \|v\|_{2(\sigma+1)}^{\sigma+1}. \end{aligned}$$

By (12), we conclude that

$$I_2 \leq C_\alpha \|v\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r. \quad (26)$$

**Case 2.**  $N \geq 3$ :

By Hölder's inequality, we have

$$\begin{aligned} I_2 &\leq \|B(|v|^{\sigma+1})\|_{r'\theta_1} \|v|^{\sigma-1}\|_{r'\beta_1} \|v_1 - v_2\|_{r'\gamma_1} \\ &= \|B(|v|^{\sigma+1})\|_{r'\theta_1} \|v\|_{(\sigma-1)r'\beta_1}^{\sigma-1} \|v_1 - v_2\|_{r'\gamma_1}, \end{aligned} \quad (27)$$

where  $\frac{1}{\theta_1} + \frac{1}{\beta_1} + \frac{1}{\gamma_1} = 1$ .

Now, by (8), (9) and (18), we have

$$\begin{aligned} \|B(|v|^{\sigma+1})\|_{r'\theta_1} &\leq C\|B(|v|^{\sigma+1})\|_{W^{2,m}} \\ &\leq C_\alpha\||v|^{\sigma+1}\|_m \\ &= C_\alpha\|v\|_{(\sigma+1)m}^{\sigma+1}, \end{aligned}$$

where  $\frac{1}{r'\theta_1} \geq \frac{1}{m} - \frac{2}{N}$  when  $\frac{1}{m} - \frac{2}{N} \geq 0$ , and  $r'\theta_1 \geq m$  when  $\frac{1}{m} - \frac{2}{N} < 0$ , and  $m > 1$  to be decided later. Then we obtain

$$I_2 \leq C_\alpha\|v\|_{(\sigma+1)m}^{\sigma+1}\|v\|_{(\sigma-1)r'\beta_1}^{\sigma-1}\|v_1 - v_2\|_{r'\gamma_1}. \quad (28)$$

Now, by requiring  $(\sigma+1)m = (\sigma-1)r'\beta_1 = r'\gamma_1 = r$  (choose  $\beta_1 = \infty$  when  $\sigma = 1$ ), we have

$$\begin{aligned} \theta_1 &= \frac{r-1}{r-1-\sigma} > 1 \Rightarrow \sigma < r-1 \\ \beta_1 &= \frac{r-1}{\sigma-1} > 1 \Rightarrow \sigma < r \\ \gamma_1 &= r-1 > 1 \\ m &= \frac{r}{\sigma+1} > 1 \\ \sigma &< \frac{N+2}{2N}r-1 \quad \text{or} \quad \sigma > \frac{r-1}{2}. \end{aligned}$$

Since  $2 < r \leq \frac{2N}{N-2}$ , we obtain  $\sigma < \frac{4}{N-2}$ , i.e.,

$$I_2 \leq C_\alpha\|v\|_{H^1}^{2\sigma}\|v_1 - v_2\|_r.$$

The same can be shown for (21). □

Now, we are ready to prove Proposition 2.

*Proof.* Recall that  $f(v) = B(|v|^{\sigma+1})|v|^{\sigma-1}v$ , by direct calculation, we have

$$\frac{\partial f(v)}{\partial v} = \frac{\sigma+1}{2}[B(|v|^\sigma)|v|^{\sigma-1}v + B(|v|^{\sigma+1})|v|^{\sigma-1}] \quad (29)$$

$$\frac{\partial f(v)}{\partial v^*} = \frac{\sigma+1}{2}B(|v|^{\sigma-1}v)|v|^{\sigma-1}v + \frac{\sigma-1}{2}B(|v|^{\sigma+1})|v|^{\sigma-3}v^2 \quad (30)$$

when  $v \neq 0$ , where  $v^*$  is the conjugate of the complex-valued function  $v$ .

For  $v = 0$ , we have

$$\begin{aligned} \frac{\partial f(v)}{\partial v} &= 0, \\ \frac{\partial f(v)}{\partial v^*} &= 0. \end{aligned}$$

Now, by Lemma 3 and Lemma 4 we obtain

$$\begin{aligned} \|\frac{\partial f(v)}{\partial v}(v_1 - v_2)\|_{r'} &\leq C_\alpha\|v\|_{H^1}^{2\sigma}\|v_1 - v_2\|_r, \\ \|\frac{\partial f(v)}{\partial v^*}(v_1 - v_2)\|_{r'} &\leq C_\alpha\|v\|_{H^1}^{2\sigma}\|v_1 - v_2\|_r. \end{aligned}$$

When  $v_1v_2 \neq 0$ , by Mean-Value Theorem, we have

$$|f(v_1) - f(v_2)| \leq \max\{|\frac{\partial f(v)}{\partial v}|, |\frac{\partial f(v)}{\partial v^*}|\}|v_1 - v_2|.$$

That is, for some intermediate point  $v_0$  between  $v_1$  and  $v_2$

$$\begin{aligned} \|f(v_1) - f(v_2)\|_r &\leq \tilde{C}_\alpha \|v_0\|_{H^1}^{2\sigma} \|v_1 - v_2\|_r \\ &\leq C_\alpha (\|v_1\|_{H^1} + \|v_2\|_{H^1})^{2\sigma} \|v_1 - v_2\|_r. \end{aligned}$$

When  $v_1 v_2 = 0$ , without loss of generality, we assume that  $v_2 = 0$ , then

$$\begin{aligned} \|f(v_1) - f(v_2)\|_{r'} &= \|f(v_1) - 0\|_{r'} \\ &= \|B(|v_1|^{\sigma+1})|v_1|^{\sigma-1}v_1\|_{r'} \\ &\leq C_\alpha (\|v_1\|_{H^1} + \|v_2\|_{H^1})^{2\sigma} \|v_1\|_r, \end{aligned}$$

where the last inequality follows from direct application of Lemma 4.

Therefore, we conclude that, for any  $v_1, v_2 \in H^1 \subset L^r$ ,

$$\|f(v_1) - f(v_2)\|_{r'} \leq C_\alpha (\|v_1\|_{H^1} + \|v_2\|_{H^1})^{2\sigma} \|v_1 - v_2\|_r.$$

□

Next, we will give some elementary properties of the free evolution (linear Schrödinger equation) formally defined by the group of operators

$$U(t) = \exp(it\Delta), \quad (31)$$

where  $t \in \mathbb{R}$ . In the following, we will state some well-known results about the operator  $U(t)$  without proving them (see, e.g., [10],[21]).

**Lemma 5.** *For any  $r \geq 2$ , and for any  $t \neq 0$ ,  $U(t)$  is a bounded linear operator from  $L^{r'}$  to  $L^r$ , and the map  $t \rightarrow U(t)$  is strongly continuous. Moreover, for all  $t \in \mathbb{R} \setminus \{0\}$ , one has*

$$\|U(t)v\|_r \leq (4\pi|t|)^{\frac{N}{r} - \frac{N}{2}} \|v\|_{r'} \quad (32)$$

for all  $v \in L^{r'}$ .

**Corollary 6.** *Let  $I$  be an interval of  $\mathbb{R}$ , and let  $v \in \mathbb{C}(I, L^{r'})$ . Then for all  $t \in \mathbb{R}$  the map  $\tau \rightarrow U(t-\tau)v(\tau)$  is continuous from  $I \setminus \{t\}$  into  $L^r$ .*

### 3. EXISTENCE AND UNIQUENESS OF LOCAL SOLUTIONS

In this section we will prove a local existence and uniqueness theorem of solutions to system (4) by a fixed point technique.

The integral equation

$$v(t) = U(t-t_0)v_0 + i \int_{t_0}^t U(t-\tau)f(v(\tau)) d\tau \quad (33)$$

may be considered as the integral version of the initial value problem for equation (4).

Defining the subspace  $Y(I) \subset \mathcal{C}(I, X)$  and  $Y_b(I) \subset \mathcal{C}_b(I, X)$  by

$$\begin{aligned} Y(I) &= \{v : v \in \mathcal{C}(I, X) \text{ and } v(t) = U(t-s)v(s) \text{ for all } s \text{ and } t \in I\} \\ Y_b(I) &= Y(I) \cap \mathcal{C}_b(I, X). \end{aligned}$$

Here for special interest we choose the Banach space  $X = L^r(\mathbb{R}^N)$ , for some  $r > 2$ , which is specified in the proof of Lemma 3 and Lemma 4, and  $\bar{X} = L^{r'}(\mathbb{R}^N)$ .

If  $v \in \mathcal{C}_b(I, X)$ , we shall denote its norm by  $|v|_I$ , and for  $v \in \mathcal{C}_b(I, H^1)$ , we denote its norm by  $|v|_{H^1, I}$ . The ball of radius  $R$  in  $\mathcal{C}_b(I, X)$  will be denoted by  $B(I, R)$ .



Let  $t_1, t_2 \in \mathbb{R}$  and let  $v(t)$  be a family of complex-valued functions defined on  $\mathbb{R}^N$ , depending on a parameter  $t \in \mathbb{R}$ . We formally define the operators

$$[G(t_1, t_2)v](t) = i \int_{t_1}^{t_2} U(t - \tau) f(v(\tau)) d\tau, \quad (34)$$

where  $f$  is the nonlinear term defined in (13). The first lemma below gives a meaning to the expression defined by (34) and contains some of its properties.

**Lemma 7.** *For any interval  $I \subset \mathbb{R}$  (possibly unbounded), the maps  $(t_1, t_2, v) \rightarrow G(t_1, t_2)v$  are continuous from  $I \times I \times \mathcal{C}(I, X)$  to  $Y_b(\mathbb{R})$ . Moreover, for any  $t_1, t_2 \in I$ , ( $t_1 < t_2$ ), for any compact sub-interval  $J$  such that  $[t_1, t_2] \subset J \subset I$ , and for any  $t \in [t_1, t_2]$ , for any  $v_1, v_2 \in \mathcal{C}(I, X)$  the  $G$  operator satisfies the estimates*

$$\|G(t_1, t_2)v_1(t) - G(t_1, t_2)v_2(t)\|_r \leq k' \|v_1 - v_2\|_J |t_2 - t_1|^{\frac{N}{r} - \frac{N}{2} + 1}$$

where  $k' = k(4\pi)^{\frac{N}{r} - \frac{N}{2}}$ ,  $k = C_\alpha (\|v_1\|_{H^1} + \|v_2\|_{H^1})$ , which is derived in the proof of Proposition 2.

*Proof.* For any  $v \in \mathcal{C}(I, X)$  the function  $\tau \rightarrow f(v(\tau))$  belongs to  $\mathcal{C}(I, \bar{X})$  as consequence of Proposition 2. Therefore, by Lemma 3, for any  $t \in \mathbb{R} \setminus \{t\}$  the function

$$\tau \rightarrow U(t - \tau)f(v(\tau)) \quad (35)$$

is continuous from  $I$  to  $X$ . To check the integrability of the function (35) it will be enough to show the integrability of its norm. More generally one is interested in the integrability of

$$\|U(t - \tau)[f(v_1(\tau)) - f(v_2(\tau))]\|_r, \quad (36)$$

for any  $v_1, v_2 \in \mathcal{C}(I, H^1) \subset \mathcal{C}(I, X)$ .

This is a direct consequence of Proposition 2 and Lemma 3: For  $t \in \mathbb{R}$ , for every compact sub-interval  $J \subset I$  and  $\tau \in J$ , we have

$$\|U(t - \tau)[f(v_1(\tau)) - f(v_2(\tau))]\|_{r'} \leq (4\pi|t - \tau|)^{\frac{N}{r} - \frac{N}{2}} k \|v_1 - v_2\|_J.$$

Finally, we come to the conclusion that

$$\|G(t_1, t_2)v_1(t) - G(t_1, t_2)v_2(t)\| \leq k' \|v_1 - v_2\|_J |t_2 - t_1|^{\frac{N}{r} - \frac{N}{2} + 1}.$$

□

Now, in order to study the equation (33) one needs the operators

$$[F(t_0)v](t) = [G(t_0, t)v](t). \quad (37)$$

The existence and properties of  $F$  follow immediately from Lemma 5.

For every  $v \in \mathcal{C}(I, H^1) \subset \mathcal{C}(I, X)$ ,

$$[A(t_0, v_0)v](t) = [F(t_0)v](t) + U(t - t_0)v_0 \quad (38)$$

is a continuous map from  $\mathcal{C}(I, H^1) \subset \mathcal{C}(I, X)$  into  $\mathcal{C}(I, X)$ .

With these notations equation (33) may be rewritten as

$$A(t_0, v_0)v = v. \quad (39)$$

The next lemma gives some elementary properties of the solutions of equation (33). In particular, it expresses the consistency of the change of the initial time  $t_0$ .

**Lemma 8.** *Let  $I$  and  $J$  be two intervals of  $\mathbb{R}$ ,  $J \subset I$ , let  $t_0 \in J$ , let  $v_0 \in H^1$  be such that the function  $t \rightarrow U(t - t_0)v_0$  belongs to  $Y(I)$ , and let  $v \in \mathcal{C}(J, X)$  be a solution of the equation (39)*

(i) *The function*

$$\phi(v) : s \rightarrow U(\cdot - s)v(s) = [\phi(v)](s) \quad (40)$$

*belongs to  $\mathcal{C}(J, Y(I))$  and satisfies for all  $s, s' \in J$  the equality*

$$[\phi(v)](s) - [\phi(v)](s') = G(s', s)v. \quad (41)$$

*Furthermore, if for some  $s \in J$ ,  $[\phi(v)](s) \in Y_b(I)$ , then  $\phi(v) \in \mathcal{C}(J, Y_b(I))$ . If in addition  $J$  is bounded, then  $\phi(v) \in \mathcal{C}_b(J, Y_b(I))$ .*

(ii) *For any  $s \in J$ ,  $u$  satisfies the equation*

$$A(s, v(s))v = v. \quad (42)$$

*Proof.* Apply the operator  $U(t - s)$  to equation  $[A(t_0, v_0)v](s) = v(s)$  and use the fact that  $U(t - s)[G(t_1, t_2)v](s) = [G(t_1, t_2)v](t)$  (for the proof of this identity, we refer to Ginibre and Velo [10]) yields

$$[\phi(v)](s) = U(\cdot - t_0)v_0 + G(t_0, s)v. \quad (43)$$

From which (41) follows immediately. The continuity properties of the left-hand side of (43) are then a consequence of the assumptions made on  $v_0$  and of Lemma 7. Finally, putting  $s' = t$  in (41) and taking the values of both members at  $t$  one obtains equation (42) at time  $t$ .  $\square$

We are now ready to discuss the problem of the existence and uniqueness of solutions of equation (39).

**Theorem 9.** *For any  $\rho > 0$ , there exists a  $T_0(\rho) > 0$ , depending only on  $\rho$ , such that for any  $t_0 \in \mathbb{R}$  and for any  $v_0 \in H^1$ , for which  $\|v_0\|_{H^1} \leq \rho$ , equation (39) has a unique solution on  $\mathcal{C}(I, X)$ , where  $I = [t_0 - T_0(\rho), t_0 + T_0(\rho)]$  and  $X = L^r$ .*

*Proof.* Let  $\rho$  be a fixed positive number, let  $t_0$  and  $T \in \mathbb{R}$ ,  $T_0 > 0$ , and let  $I = [t_0 - T, t_0 + T]$ . Then for every  $v_1, v_2 \in H^1$  and  $\|v_1 - v_2\|_{H^1} \leq 2\rho$ , Lemma 5 and (37) yield the inequality

$$|F(t_0)v_1 - F(t_0)v_2|_I \leq 2k'|t - t_0|^{\frac{N}{r} - \frac{N}{2} + 1}|v_1 - v_2|_I. \quad (44)$$

In particular, if we take  $T = T_0(\rho)$  with  $T_0(\rho)$  defined by

$$4k'|T_0(\rho) - t_0|^{\frac{N}{r} - \frac{N}{2} + 1} = 1 \quad (45)$$

in equality (44) it gives

$$|F(t_0)v_1 - F(t_0)v_2|_I \leq \frac{1}{2}|v_1 - v_2|_I. \quad (46)$$

Let now  $v_0 \in X$  be such that  $U(\cdot - t_0)v_0 \in B(I, \rho)$ . Definition (38) and estimate (46) imply

$$|A(t_0, v_0)v|_I \leq 2\rho, \quad (47)$$

and

$$|A(t_0, v_0)v_1 - A(t_0, v_0)v_2|_I \leq \frac{1}{2}|v_1 - v_2|_I, \quad (48)$$

for all  $v, v_1, v_2 \in B(I, 2\rho)$ , from which it follows that  $A(t_0, v_0)$  is a contraction from the ball  $B(I, 2\rho)$  into itself. The result is now a consequence of the contraction mapping theorem.  $\square$

## 4. GLOBAL EXISTENCE OF SOLUTIONS

In this section we will study global existence of solutions to system (4) under the condition of  $\sigma \geq 1$ . We will show below that we have global solutions when  $1 \leq \sigma < \frac{4}{N}$ . Comparing this to the results of the classical NLS ( $\sigma < \frac{2}{N}$ ), we “gain” global regularity for larger range of values of  $\sigma$ . As we stated in the introduction, system (4) will recover the classical NLS as the parameter  $\alpha \rightarrow 0^+$ , in a subsequential paper, we will study numerically the blow up profile of the classical NLS by focusing on system (4) with  $\frac{2}{N} \leq \sigma < \frac{4}{N}$  when  $N \leq 3$ . To be more specific, the profile of blow-up in the critical case  $\sigma = \frac{2}{N}$  in the classical NLS has not been known completely, in a subsequential work, we will compute SH system (4) and try to find out the blow up profile by forcing the parameter  $\sigma$  to approach zero.

**Theorem 10.** *Let  $v_0 \in H^1(\mathbb{R}^N)$ . If  $1 \leq \sigma < \frac{4}{N}$ , then there exists a unique solution  $v \in C((-\infty, \infty); H^1(\mathbb{R}^N))$  of the initial-value problem (4), in the sense of the equivalent integral equation. Furthermore, as long as  $v(x, t)$  remains in  $H^1(\mathbb{R}^N)$ , the energy*

$$\mathcal{N}(v) = \int_{\mathbb{R}^N} |v(x, t)|^2 dx \quad (49)$$

and Hamiltonian

$$\mathcal{H}(v) = \int_{\mathbb{R}^N} \left( |\nabla v(x, t)|^2 - \frac{u(x, t)|v(x, t)|^{\sigma+1}}{\sigma+1} \right) dx \quad (50)$$

remain constant in time.

In the local existence theorem in section 3, we have shown that the length  $T_0$ , of the interval of existence  $[t_0, t_0 + T_0]$ , can be taken to depend only on  $\|v_0\|_{H^1}$ . It follows that if  $v(x, t)$  is a maximally defined solution on  $[t_0, T_{\max})$ , then either

$$T_{\max} = +\infty$$

or

$$\lim_{t \rightarrow T_{\max}^-} \|v(t)\|_{H^1} = +\infty.$$

The heart of the global existence proof lies in the use of the invariants (49) and (50), which enable us to obtain an *a priori* bound of the following type:

$$\|v(x, t)\|_{H^1} \leq C(\mathcal{N}, \mathcal{H}). \quad (51)$$

*Proof.* We proceed as follows:

From (50) we have

$$\|\nabla v(x, t)\|^2 \leq \mathcal{H} + \frac{1}{\sigma+1} \int_{\mathbb{R}^N} u|v|^{\sigma+1} dx. \quad (52)$$

Observe that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u|v|^{\sigma+1} dx \right| &\leq \|u\|_p \|v|^{\sigma+1}\|_{p'} \\ &= \|u\|_p \|v\|_{p'(\sigma+1)}^{\sigma+1} \end{aligned} \quad (53)$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p, p' < \infty.$$

**Case 2.**  $N \leq 2$ :

By (8) and (18), we have

$$\begin{aligned} \|u\|_p &\leq C\|u\|_{W^{2,p}} \\ &= C\|B(|v|^{\sigma+1})\|_{W^{2,p}} \\ &\leq C_\alpha\| |v|^{\sigma+1} \|_p \\ &= C_\alpha\|v\|_{(\sigma+1)p}^{\sigma+1} \end{aligned}$$

for any  $p > 1$ .

Plug the above inequality into (53) and requiring  $p = p' = 2$ , we obtain

$$\left| \int_{\mathbb{R}^N} u|v|^{\sigma+1} dx \right| \leq C_\alpha \|v\|_{2(\sigma+1)}^{2(\sigma+1)}.$$

By (7), taking  $q = 2(\sigma + 1)$ , we obtain

$$\|v\|_{2(\sigma+1)} \leq C\|v\|^{1-\frac{2(\sigma+1)-2}{2 \cdot 2(\sigma+1)}N} \|\nabla v\|^{\frac{2(\sigma+1)-2}{2 \cdot 2(\sigma+1)}N},$$

with

$$0 < \frac{2(\sigma+1)-2}{2 \cdot 2(\sigma+1)}N < 1,$$

which is always satisfied when  $N \leq 2$ .

Then (52) yields

$$\|\nabla v(t)\|^2 \leq \mathcal{H}_0 + C_\alpha \|v_0\|^{2(\sigma+1)-(\sigma N-2)} \|\nabla v\|^{\sigma N-2}. \quad (54)$$

**Case 2.**  $N \geq 3$ :

By (9) and (18), we obtain

$$\begin{aligned} \|u\|_p &\leq C\|u\|_{W^{2,m}} \\ &= C\|B(|v|^{\sigma+1})\|_{W^{2,m}} \\ &\leq C_\alpha\| |v|^{\sigma+1} \|_m \\ &= C_\alpha\|v\|_{(\sigma+1)m}^{\sigma+1}, \end{aligned}$$

where

$$\frac{1}{p} = \frac{1}{m} - \frac{2}{N} > 0 \Rightarrow m < \frac{N}{2}. \quad (55)$$

Plugging into (53) and requiring  $m = p'$ , i.e.,  $m = \frac{2N}{N+2}$ , we get

$$\left| \int_{\mathbb{R}^N} u|v|^{\sigma+1} dx \right| \leq C_\alpha \|v\|_{m(\sigma+1)}^{2(\sigma+1)}.$$

By (7), taking  $q = m(\sigma + 1)$ , we obtain

$$\|v\|_{m(\sigma+1)} \leq C\|v\|^{1-\frac{m(\sigma+1)-2}{2m(\sigma+1)}N} \|\nabla v\|^{\frac{m(\sigma+1)-2}{2m(\sigma+1)}N},$$

with

$$0 < \frac{m(\sigma+1)-2}{2m(\sigma+1)}N < 1 \Rightarrow \sigma < \frac{4}{N-2}. \quad (56)$$

Then (52) yields

$$\|\nabla v(t)\|^2 \leq \mathcal{H}_0 + C_\alpha \|v_0\|^{2(\sigma+1)-(\sigma N-2)} \|\nabla v\|^{\sigma N-2}. \quad (57)$$

For (54) and (57),  $\|\nabla v\|$  is bounded when  $\sigma N - 2 < 2$ , i.e.,  $\sigma < \frac{4}{N}$ . Therefore, the  $H^1$  norm of the solution  $v$  is bounded uniformly independent of time  $t$ , so we can conclude that we have global solution for any  $1 \leq \sigma < \frac{4}{N}$ .  $\square$

In conclusion, we have shown that the Schrödinger-Newton system (3) and the Schrödinger-Helmholtz system (4) admit short time unique solution when  $1 \leq \sigma < \frac{4}{N-2}$  (by definition,  $\frac{4}{N-2} = \infty$  when  $N \leq 2$ ) and global existence of unique solution when  $1 \leq \sigma < \frac{4}{N}$ . Comparing to the result of classical NLS (1) ( $\sigma < \frac{2}{N-2}$  for local existence and  $\sigma < \frac{2}{N}$  for global existence), one expects this “better” result since the nonlinear terms in system (3) and system (4) are milder than that of the classical nonlinear Schrödinger equation (1).

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